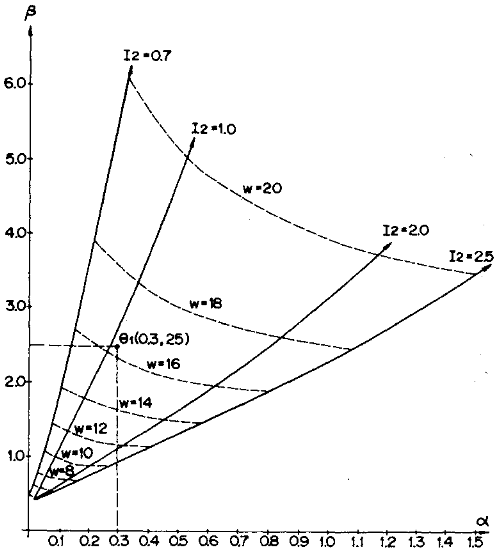

Fig. 2 Limit-cycle loci of a rocket control system.

Table 2 Limit-cycle values

Order of structure	Frequency (rad/sec)	Amplitudes I_1	I_2
original	15.7	1.34	1.04
5th	15.68	1.25	0.97
4th	17.4	1.32	0.54
3rd	15.75	1.33	1.05
2nd	15.65	1.26	0.97

Table 3 Limit-cycle values

Order of TF_{str}	Frequency (rad/sec)	Amplitudes I_1	I_2
original	64	1.20	1.32
5th	40	1.62	1.001
3rd	46.6	1.605	0.318
2nd	12.55	1.19*	0.32*

Table 4 Limit-cycle values

Order of TF_{str}	Frequency (rad/sec)	Amplitudes I_1	I_2
original	64	1.2	1.32
5th	13.55	1.31	1.67
4th	14.3	1.24	1.04
3rd	13.7	1.36	1.37
2nd	13.2	1.39	1.47

It can be seen that the structure transfer function can be replaced by its lower-order models without producing appreciable errors on limit cycles.

Control System with Low Damping Ratio

From Ref. 3, the original transfer function of the structure filter is

$$G_{sf}(S) = \frac{(S^2+12S+5810)(S^2+22S+12800)}{(S^2+13S+3520)(S^2+20.8S+21000)} \tag{18}$$

With this structure filter, the damping ratio of the structure loop is very low.[4] Using the same method as in the previous section, the simulated results for the structure loop are given in Table 3. The asterisk again represents the magnitude of the fundamental component of a limit cycle. For the overall system (with motor-linkage gain equal to 0.3), the simulated results are given in Table 4.

As the gain of the motor-linkage was changed to 0.1, a limit cycle was found at $\omega = 15$ and $I_2 = 4.34$. This result has been checked by computer simulations with both the original model and the reduced models. From these results, it can be seen that the analyses by use of the low-order models give correct results only if the limit-cycle frequency is low and the system damping is large.

Conclusions

For limit-cycle analysis of a nonlinear rocket control system, it has been shown that the reduced models can give good approximation to the original system only if the frequency of the limit cycle is low and the damping of the system is large. Since the frequency of a limit cycle can be found only after the analysis is completed, it is, therefore, advisable to use the original transfer functions instead of the reduced ones for limit-cycle analysis of the nonlinear rocket control system.

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Temperature Distribution in a Sublimation-Cooled Coated Cylinder in Convective and Radiative Environments

K. N. Shukla*
Vikram Sarabhai Space Centre, Trivandrum, India

Nomenclature

a	= thermal diffusivity coefficient
Bi	= Biot criteria
c	= heat capacity
Fo	= Fourier number
h	= heat transfer coefficient
ΔH	= enthalpy rate
$J_k(x)$	= Bessel function of first kind of order k on argument x.
K	= thermal conductivity coefficient
$\dot{m}$	= transpiration or sublimation rate
r	= radial coordinate
R	= Radius of the cylinder
Sk	= Stark number
$T(r,t)$	= temperature distribution
t	= time variable
x	= dimensionless radial coordinate

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*Scientist, Physics and Applied Mathematics Division.

$Y_k(x)$ = Bessel function of second kind of order k on argument x.
α, β = insulation temperature dependent coefficients
ρ = specific density of the material
σ = Boltzman's constant
θ = dimensionless temperature distribution
ϕ = dimensionless simplex

Subscripts

i = 1, 2
j = 1, 2, 3, ...
n = 1, 2, 3, ..., ∞

Introduction

AMONG the numerous problems posed by the design of a space vehicle, one of the more challenging is the high-temperature thermal protection of the vehicle. The temperature of the body surface is highly elevated during high-speed flight, which can cause structural failures and damage of the electronic equipment. As a safety measure, vehicle designs incorporate thermal coating of the payload compartment to keep it below a design temperature level to guarantee the correct functioning of the equipment. Efficient design of such systems require complete knowledge of the temperature histories of the coating and metal skin.

The use of transpiration or sublimation cooled transparent coating material has received much attention. Howe[1] and Peterson et al.[2] have performed studies on this type of coating material. In another study, Howe and Green[3] have analyzed sublimation cooled coated-surface exposed in convective and radiative environments.

In the present analysis, the thermal response of a sublimation cooled coated cylindrical body is considered. The surface of the body is exposed in convective and radiative environments and temperature histories of the coating and metal skin are determined.

Analysis

Consider a cooled cylindrical body of radius R_1, having a semitransparent coating of outer radius R_2. The system behaves like a composite body. Assuming constant material properties, the Fourier heat conduction equation can be written as

$$\rho_i c_i \frac{\partial T_i}{\partial t} = \frac{K_i}{r}\frac{\partial}{\partial r}\left[r\frac{\partial T_i}{\partial r}\right], t>o \tag{1}$$

An energy balance at the exposed face yields

$$\left[K_2\frac{\partial T_2}{\partial r}\right]_{r=R_2} = h(T_a-T_2)-\epsilon\sigma T_2^4-\dot{m}\Delta H \tag{2}$$

where the first term on the right hand side is due to convective heat flux incident uniformly on the coating, the second due to surface radiation and the last due to cooling or sublimation of an embedded solid material in the coating matrix. The other boundary conditions are the continuity requirements at the interface, i.e.

$$T_1(R_1,t)=T_2(R_2,t) \tag{3a}$$

and

$$K_1\frac{\partial T_1(R_1,t)}{\partial r}=K_2\frac{\partial T_2(R_2,t)}{\partial r} \tag{3b}$$

Further, let this initial temperature distribution of the system be uniform

$$T_i(r,o)=T_{io} \tag{4}$$

For convenience, introduce the following normalized variations in the analysis:

$$x=\frac{r}{R},\ \theta_i=\frac{T_i}{T_a},\ Fo=\frac{a_i t}{R_1^2},$$

$$Bi=\frac{hR_1}{K_2},\ K_\gamma=\frac{K_1}{K_2},\ K_a^2=\frac{a_1}{a_2}$$

$$K_R=\frac{R_2}{R_1},\ Sk=\frac{\epsilon\sigma T_a^3 R_2}{K_2},\ \phi=\frac{\dot{m}\Delta H R_2^2 T_a}{K_2}$$

The system of Eq. (1) reduces to

$$\frac{\partial\theta_i}{\partial Fo}=\frac{M_i}{x}\frac{\partial}{\partial x}\left[x\frac{\partial\theta_i}{\partial x}\right],\ M_1=1,\ M_2=1/K_a^2 \tag{5}$$

with boundary conditions

$$\frac{\partial\theta_2}{\partial x}-Bi(1-\theta_2)+Sk\theta_2^4+\phi=o,\ x=K_R \tag{6}$$

$$\theta_1(1,Fo)=\theta_2(1,Fo) \tag{7}$$

$$K_\lambda\frac{\partial\theta_i(1,Fo)}{\partial x}=\frac{\partial\theta_2(1,Fo)}{\partial x} \tag{8}$$

and the initial conditions

$$\theta_i(x,o)=\theta_{io} \tag{9}$$

The system of Eqs. (5-9) form a coupled nonlinear boundary value problem which is difficult to solve. Hence some approximate methods for the analytical treatment must be worked out to extract a meaningful solution. As treated, the linearization of Eq. (6) is appropriate[4] if Sk is small, say $0\le Sk\le 0.8$ and $\theta_{20}\ge 0.2$, but the method fails for $Sk>0.8$. It has been suggested[5] that θ_2^4 (1, Fo) can be approximated by a linear function

$$\theta_2^4(1,Fo)\approx-\alpha+\beta\theta_2(1,Fo) \tag{10}$$

where the coefficients α and β depend on the temperature of the exposed surface.

Assuming that it is required to find the temperature field $\theta(x,Fo)$ for some arbitrarily specified time instant $Fo=Fo$, α and β can be obtained from

$$-\alpha+\beta\theta_2^*=\theta_2^{*4} \tag{11}$$

and

$$\int_{\theta_{20}}^{\theta_2^*}(-\alpha+\beta\theta_2)\,d\theta=\int_{\theta_{20}}^{\theta_2^*}\theta^4 d\theta \tag{12}$$

where

$$\theta_2^*=\theta_2(1,Fo^*)$$

From the above

$$\beta=2\left[\frac{\theta_2^{*4}}{\theta_2^*-\theta_{20}}-\frac{\theta_2^{*5}-\theta_{20}^5}{5(\theta_2^*-\theta_{20})^3}\right] \tag{13}$$

On substituting the above approximation for θ_2^4 into the nonlinear boundary condition (6), we have

$$\frac{\partial\theta_2}{\partial x}+S_k^*(\theta_2-\theta_2^\circ)=0,$$

$$\theta_2^\circ=(Bi+\alpha Sk-\phi)/S_k^*,\ S_k^*=Bi+\beta Sk \tag{14}$$

The system of differential Eqs. (5) under the boundary conditions (7,8,14) and the intial conditions (9) can be solved by Laplace transform technique. The solutions are

$$\frac{\theta_1-\theta_{10}}{\theta_{20}-\theta_{10}} = 1-L-2\sum_{n=1}^{\infty} A_n\, J_o(\mu_n x)\, \exp(-\mu_n^2 Fo) \tag{15}$$

and

$$\frac{\theta_2-\theta_{20}}{\theta_{20}-\theta_{10}} = -L-2\sum_{n=1}^{\infty}[B_{n1}J_o(\mu_n x) + B_{n2}Y_o(\mu_n x)] \exp(-\mu_n^2 Fo) \tag{16}$$

where

$$L=(\theta_{20}-\theta_2^\circ)\ /\ (\theta_{20}-\theta_{10})$$

and

$$A_n = \frac{\nu_n}{\pi_n}[p_{n2}J_1(\nu_n)+P_{n1}Y_1(\nu_n)-LS_k^*\{J_o(\nu_n)Y_1(\nu_n) + J_1(\nu_n)Y_0(\nu_n)\}\}],$$

$$B_{n1} = \frac{\nu_n}{\pi_n}[K_bP_{n2}J_1(\mu_n)-LS_k^* Q_{n1}]$$

$$B_{n2} = \frac{\nu_n}{\pi_n}[-K_bP_{n1}J_1(\mu_n) + LS_k^* Q_{n2}]$$

$$\begin{aligned}\pi_n = {} & \mu_n^2K_R[D_{n2}Q_{n1} + D_{n1}Q_{n2}] \\ & + P_{n1}[\mu_n\nu_n\{(K_a+K_b)J_0(\mu_n)Y_0(\nu_n) \\ & -(1+K_aK_b)J_1(\mu_n)Y_1(\nu_n)\} \\ & -K_a\mu_nJ_0(\mu_n)J_0(\mu_n)Y_1(\nu_n)-K_b\nu_nJ_1(\mu_n)Y_0(\nu_n)] \\ & + P_{n2}[\mu_n\nu_n\{(K_a-K_b)J_0(\mu_n)J_0(\nu_n) \\ & -(1+K_aK_b)J_1(\nu_n)J_1(\mu_n)\} \\ & -\mu_nK_aJ_0(\mu_n)J_1(\nu_n) + K_b\nu_nJ_1(\mu_n)J_0(\nu_n)]\end{aligned}$$

$$\begin{aligned} p_{n1} &= \nu_nJ_1(\nu_nK_R) - -S_k^*J_0(\nu_nK_R),\ Q_{n1}=J_0(\mu_n)J_1(\nu_n) \\ &\quad -K_bJ_1(\mu_n)Y_0(\nu_n) \\ p_{n2} &= \nu_nY_1(\nu_nK_R)-S_k^*Y_0(\nu_nK_R),\ Q_{n2}=J_0(\mu_n)Y_1(\nu_n) \\ &\quad + K_bJ_1(\mu_n)Y_0(\nu_n) \\ D_{n1} &= \nu_nJ_0(\nu_nK_R) + S_k^*J_1(\nu_nK_R),\ D_{n2}=S_k^*Y_1(\nu_nK_R) - \\ &\quad -\nu_nY_0(\nu_nK_R)\end{aligned}$$

The characteristic roots μ_n and ν_n are related to each other by the relation

$$\mu_n = k_a\nu_n \tag{17}$$

and they are obtained from the equation

$$p_{n2}Q_{n1} + p_{n1}Q_{n2} = 0 \tag{18}$$

Equations (15) and (16) give the general solution for the temperature distribution in the metal skin and the coating material. However these distributions contain the terms α and β which are still unknown because the temperature $\theta_2(1, Fo^*)$ is not determined. If we apply the method of successive approximations, the jth iteration yields the values of coefficients α and β as

$$\alpha_j = \beta_j\theta_j^* - \theta_j^{*4},\ \cdot\ \beta_j = 2\left[\frac{\theta_j^{*4}}{\theta_j^*-\theta_0} - \frac{1}{5}\ \frac{\theta_j^{*5}-\theta_0^5}{(\theta_j^*-\theta_0)^2}\right] \tag{19}$$

Table 1 Material properties

	ρgm/cc	c	kcal/sec-cm^{oc}
Silicon steel	7.77	0.11	0.097
Fused quartz	2.0	0.20	0.0023

$$\theta_j = 0.5[\theta_{j-1}(1, Fo^*) + \theta_{j-2}(1, Fo^*)] \tag{20}$$

In this way, the sequence of functions

$$\theta_{i1}(x, Fo^*),\quad \theta_{i2}(x, Fo^*) \tag{21}$$

are determined. These functions converge rapidly and approach certain limiting temperature fields $\theta_{i_{lim}}(x, Fo^*)$.

Examples

Consider a steel rod of radius 40 cm coated with fused quartz. The outer radius of the coating is 41 cm. The required material properties of the system are shown in Table 1.

Thus $K_\lambda = 42.17$, $Ka = 19.28$, and $K_R = 1.025$. In addition, we take $Bi = 16.5$, $\phi = 1.0$, and $Sk = 1.0$. Numerical calculations are performed for the temperature histories of the metal skin and coating material. For simplicity, assume that the intial temperatures of the metal skin and the coating material are equal to zero. This assumption simplifies the expressions (15) and (16) as

$$\theta_1 = \theta_2^\circ[1+2S_k^*\sum_{n=1}^{\infty}\frac{\nu_n}{\pi_n}\{J_0(\nu_n)Y_1(\nu_n) + J_1(\nu_n)Y_0(\nu_n)\}J_0(\mu_n x)\exp(-\mu_n^2Fo)] \tag{22}$$

$$\theta_2 = \theta_2^\circ[1-2\sum_{n=1}^{\infty}\frac{\nu_n}{\pi_n}\{Q_{n1}J_0(\mu_n x) - Q_{n2}Y_0(\mu_n x)\}\exp(-\mu_n^2Fo)] \tag{23}$$

From the analysis of the characteristic equation (18), it is observed that there is an infinite number of the characteristic roots occurring in the ascending order, $\mu_1<\mu_2<\mu_3$... Further, the expressions (22) and (23) containing the exponentially decreasing series in Fo, converges rapidly with the generalized time Fo. Thus it is appropriate to consider only first term μ_1 to find a sufficiently accurate result.

The computed results show that the functions are rapidly convergent and five to six iterations of θ_{ik} are sufficient to give limiting temperature field $\theta_{i_{lim}}(x, Fo^*)$. It is also found that the temperature of the coating surface attains the steady state almost immediately (Fig. 1).

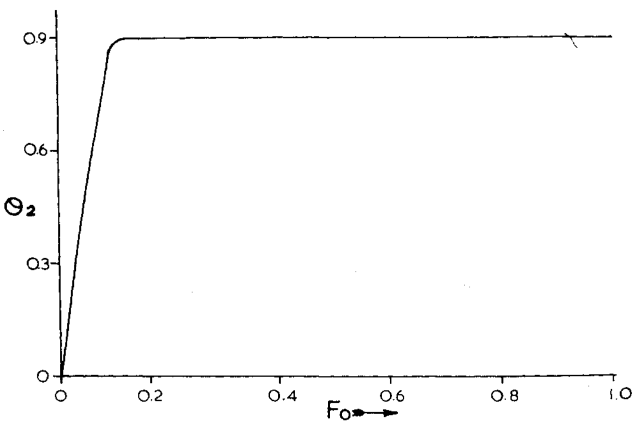

Fig. 1 Temperature history of coating surface (θ_2 vs Fo).

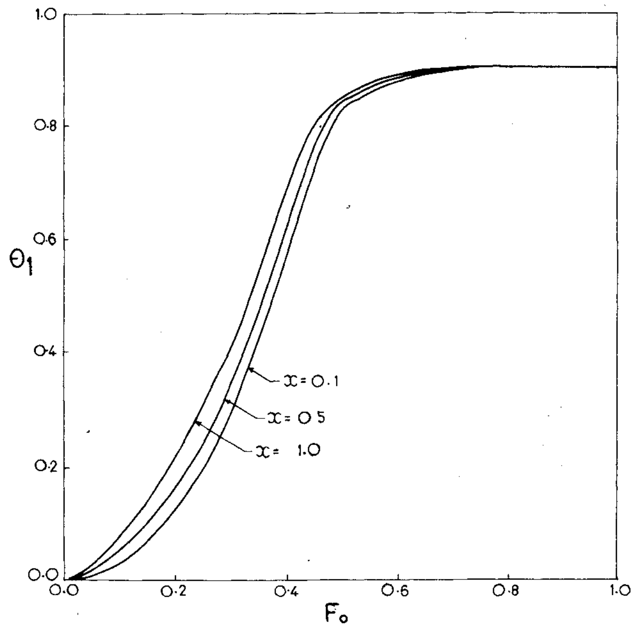

Fig. 2 Temperature profile in metal skin (θ_l vs Fo/β).

Figure 2 shows the build up of temperature profile in the metal skin at different points (for $x = 0.1$, 0.5, 1.0). For $Fo < 0.4$, there is a sharp variation in θ_l and afterwards it establishes a thermodynamic equilibrium in the metal skin and the coating material.

Conclusion

The effectiveness of the method proposed by Ivanov and Medvedev dealing with complex problems are shown. Analytical solutions for the temperature histories in the metal skin and the protective coating are determined and it has been observed that coatings subjected to radiative and convective heating can withstand fairly well in a steady state. This analysis may help designers to select proper coating materials and suitable boundary conditions to keep the payload compartment below a design temperature level.

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